Honours Analysis
Course Notes

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1 Sequences and Series of Functions: Uniform Convergence and Power Series

1.1 Uniform Convergence of Sequences

Definition 1.1 (Pointwise Convergence)**.** Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to converge pointwise on E, as $n \to \infty$, iff for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well as ϵ) such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

Remark 1.2. pointwise convergence does not necessarily guarantee preservation of continuity, integration or Differentiability:

- Continuity and differentiability example: $f_n(x) = x^n$ and set $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$
- differentiable f_n and f but not equal limits: $f_n = x^n/n$ and $f(x) = 0$. Then $f_n'(x) = x^{n-1} = 1$ when $x = 1$ but $f'(1) = 0$.
- integrable: $f_n(x) = \begin{cases} 1, & x = p/m \in \mathbb{Q}, m \leq n \\ 0 & \text{otherwise} \end{cases}$ and $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

Definition 1.3 (Uniform Convergence)**.** Let $E \subseteq \mathbb{R}$, nonempty. A sequence of functions $f_n : E \to \mathbb{R}$ is said to converge uniformly on E to a function f iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

for all $x \in E$.

Example 1.4. Prove that $x^n \to 0$ uniformly on $[0, b] < 1$, and pointwise, but not uniformly on $[0, 1)$

Proof. $x^n \to 0$ pointwise on $[0, 1)$ by the following. For $x \in [0, 1)$, the sequence x^n is monotone decreasing, as $x < 1 \implies x^{n+1} < x^n$. Further, it is bounded below by 0. So, by monotone convergence theorem, this sequence has a limit. To find this limit, consider

$$\begin{aligned} x^{n+1} &= x \cdot x^n \\ \implies \lim_{n\to\infty} x^{n+1} &= x \lim_{n\to\infty} x^n \\ \implies L = xL \implies x &= 1 \text{ (contradiction), or } L = 0. \end{aligned}$$

To prove x^n does not converge uniformly on $[0,1)$, suppose it does. Then given $1/2 > \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x^N| < \epsilon$ for all x. But $x^N \to 1$ as $x \to 1$, so we can always choose an $x_0 \in (0,1)$ such that $x_0^N > \epsilon$, which is a contradiction.

To prove that it converges uniformly on $[0,b], b < 1$, let $b < 1$, and take $\epsilon > 0$. Then choose $N \in \mathbb{N}$ such that $n \geq N$ implies $b^n < \epsilon$. Then $x \in [0,b] \implies |x^n| \leq b^n < \epsilon$. □

Theorem 1.5. Uniform Convergence preserves continuity. Formally, let $E \subseteq \mathbb{R}$, nonempty, and let $f_n \to f$ uniformly on E as $n \to \infty$. If each f_n is continuous at some $x_0 \in E$, then f is also continuous at x_0.

Proof. We know that, for all $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|f(x) - f_n(x)| < \epsilon/3$ for all x. We also know that $\forall \epsilon > 0$, for each x_0 there exists a $\delta > 0$ such that $|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \epsilon/3$. Consider that, given $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

□

Theorem 1.6. Uniform convergence preserves integration. Formally, suppose $f_n \to f$ uniformly on a closed interval [a,b]. If each f_n is integrable on [a,b], then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$$

Proof. By the next Theorem, f is bounded on $[a,b]$. To prove that f is integrable, let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$n \geq N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3(b-a)}$$

for all $x \in [a,b]$. Using this equality for $n = N$, by the definition of upper and lower sums,

$$U(f - f_n, P) \leq \max(f - f_n)(b-a) \leq \frac{\epsilon(b-a)}{3(b-a)} \leq \epsilon/3$$

and

$$L(f - f_n, P) \geq \min(f - f_n)(b-a) \geq -\frac{\epsilon(b-a)}{3(b-a)} \geq -\epsilon/3$$

for any partition P of $[a,b]$. Since f_N is integrable, choose a partition P such that

$$U(f_n, P) - L(f_n, P) < \epsilon/3.$$

It follows that

$$U(f,P) - L(f,P) \leq U(f - f_n, P) + U(f_n, P) - L(f_n, P) - (f - f_n, P) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

that is, f is integrable on $[a,b]$. Then

$$\left| \int_a^x f_n(t)dt - \int_a^x f(t)dt \right| \leq \int_a^x |f_n(t) - f(t)|dt \leq \frac{\epsilon(x-a)}{3(b-a)} < \epsilon$$

for all $x \in [a,b]$ and $n \geq N$. □

Definition 1.7. A sequence of functions (f_n) is uniformly bounded on a set E iff there is an $M > 0$ such that $f_n(x) \leq M$ for all $x \in E$, and all $n \in \mathbb{N}$.

Theorem 1.8. Suppose for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ is bounded. If $f_n \to f$ uniformly on E, as $n \to \infty$, then (f_n) is uniformly bounded on E and f is a bounded function on E

Proof. We will first prove f is a bounded function on E. As $f_n \to f$ uniformly, on E, as $n \to \infty$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|f_n(x) - f(x)| \leq 1$$

for all $x \in E$.

Consider when $n = 1, 2, ...N$. Each f_n is bounded by assumption. Take the maximum of the set of minimum upper bounds from each f_n, M. Then for all n, $1 \leq n \leq N$, $|f_n| \leq M$. Combining these two facts, by the triangle inequality,

$$|f(x) = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M.$$

So, f is bounded. Now we wish to show that (f_n) is uniformly bounded. Consider $n \geq N$:

$$|f_n(x)| \leq |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + (1 + M) = 2 + M$$

So, (f_n) is uniformly bounded. □

Note: if $f_n \to f$ and $g_n \to g$ uniformly, then $f_n + g_n \to f + g$ uniformly and $f_n g_n \to fg$ uniformly.

Lemma 1.9 (Uniform Cauchy Criterion)**.** Let E be a nonempty subset of $\mathbb{R}$ and let $f_n : E \to \mathbb{R}$ be a sequence of functions. Then f_n converges uniformly on E iff for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$.

Proof. Suppose $f_n \to f$ uniformly on E as $n \to \infty$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon/2$$

for all $x \in E$. Then $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon$.

Now suppose that f_n is Cauchy for each $x \in E$. By Cauchy's theorem of sequences f_n is pointwise convergent, that is,

$$f(x) = \lim_{m\to\infty} f_m(x)$$

exists for each $x \in E$. So,

$$\lim_{m\to\infty} |f_n - f_m(x)| \leq \lim_{m\to\infty} \epsilon/2 \implies |f_n(x) - f(x)| < \epsilon.$$

□

Important: Uniformly convergent implies Cauchy, but Cauchy does not imply uniformly convergent! Unless you are working with a complete metric space (more on that later).

Useful Proposition:

Proposition 1.10. The following are equivalent for a sequence of functions $f_n : E \to \mathbb{R}$ and $f : E \to \mathbb{R}$:

- $f_n \to f$ uniformly on E.
- $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$
- there exists a convergent sequence $a_n > 0$ such that $|f_n(x) - f(x)| \leq a_n$ for all $x \in E$.

1.2 Uniform Convergence of Series

Definition 1.11. Let f_k be a sequence of functions defined on some set E and set

$$S_n(x) := \sum_{k=1}^{n} f_k(x)$$

for $x \in E, n \in \mathbb{N}$.

i) The series $\sum_{k=1}^{n} f_k(x)$ is said to converge pointwise on E iff the sequence S_n converges pointwise on E as $n \to \infty$.

ii) The series $\sum_{k=1}^{n} f_k(x)$ is said to converge uniformly on E iff the sequence S_n converges uniformly on E as $n \to \infty$.

iii) The series $\sum_{k=1}^{n} f_k(x)$ is said to converge absolutely (pointwise) iff $\sum_{k=1}^{n} |f_k(x)|$ converges pointwise for each $x \in E$.

Theorem 1.12. Let $E \subseteq \mathbb{R}$, nonempty, and let (f_k) be a sequence of real functions defined on E.

i) Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{n} f_k(x)$ converges uniformly on E, then f is continuous at $x_0 \in E$.

ii) Suppose that $E = [a, b]$, and that f_k is integrable on $[a, b]$. If $f = \sum_{k=1}^{n} f_k(x)$ converges uniformly on $[a, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \int_a^b f_k(x).$$

iii) Suppose E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{n} f_k(x)$ converges at some $x_0 \in E$, and $\sum_{k=1}^{n} f_k'(x)$ converges uniformly on E, then $f = \sum_{k=1}^{n} f_k(x)$ converges uniformly on E, f is differentiable on E and

$$\left(\sum_{k=1}^{n} f_k(x)\right)' = \sum_{k=1}^{n} f_k'(x)$$

for all $x \in E$.

Proof. These follow simply by the uniform convergence properties of sequences. □

Theorem 1.13 (Weierstrass M-test)**.** Let $E \subseteq \mathbb{R}$, nonempty, let $f_k : E \to \mathbb{R}$, and suppose that $M_k \geq 0$ satisfies $\sum_{k=1}^{\infty} M_k < \infty$. if $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on E.

Proof. Let $\epsilon > 0$. Every convergent sequence is Cauchy, and so we can choose $m \geq n \geq N$ such that $\sum_{k=n}^{m} < \epsilon$. Then

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \epsilon$$

Thus, the partial sums of $\sum_{k=1}^{\infty} f_k$ and $\sum_{k=1}^{\infty} |f_k|$ are uniformly Cauchy, and therefore by the Cauchy Criterion they are uniformly and absolutely convergent. □

Theorem 1.14 (Dirichlet's Test for Uniform Convergence)**.** Let $E \subseteq \mathbb{R}$, nonempty, and suppose that $f_k, g_k : E \to \mathbb{R}$, $k \in \mathbb{N}$. If

$$\left|\sum_{k=1}^{n} f_k(x)\right| \leq M < \infty$$

for $n \in \mathbb{N}$ and $x \in E$, and if $g_k \downarrow 0$ uniformly on E as $k \to \infty$¡ then $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

1.3 Power Series

Definition 1.15. A power series is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. We call a_n the coefficients of the power series, and c the centre of the power series.

Example 1.16. Deduce where the power series $\sum_n n^n|x|^n$ is convergent.

Proof. The sequence $n^n|x|^n$ very quickly becomes unbounded: As soon as $n \geq 2/|x|$, we have that $n^n|x|^n > 2^n$. So the only place where this sequence can converge is when $x = 0$. □

Example 1.17. Deduce where the power series $\sum_n x^n/n!$ is convergent.

Proof. By the ratio test, the sum converges absolutely for all x (has infinite radius). □

Exercise 1.18. If $(a_n r^n)$ is bounded and $0 \leq s \leq r$, then $(a_n s^n)$ is bounded.

Proof. By definition of boundedness, $\exists M \in \mathbb{R}$ such that $|a_n r^n| \leq M$ for all $n \in \mathbb{N}$. But $0 \leq s \leq r$, so $|a_n s^n| \leq |a_n r^n| \leq M$ for all $n \in \mathbb{N}$, and so $(a_n s^n)$ is bounded. □

Exercise 1.19. If $(a_n r^n)$ is unbounded and $r < s$, then $(a_n s^n)$ is unbounded.

Proof. $\forall k \in \mathbb{R}$, $\exists n_k \in \mathbb{N}$ such that $|a_{n_k} r^{n_k} > k$. But $s > r$, so $|a_{n_k} s^{n_k}| > |a_{n_k} r^{n_k}| > k$. So $(a_n s^n)$ is unbounded. □

Definition 1.20. The radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is given by

$$R := \sup\{r \geq 0 \mid (a_n r^n) is bounded\}$$

- $r < R \implies (a_n r^n)$ is bounded.
- $r > R \implies (a_n r^n)$ is unbounded.

Theorem 1.21. Suppose the radius of convergence R of a power series satisfies $0 < R < \infty$. If $|x - c| < r$, then the power series converges absolutely. If $|x - c| > R$, the power series diverges.

Proof. Let us first suppose that $|x - c| < R$. Consider a number ρ such that $|x - c| < \rho < R$. Then the sequence $(a_n \rho^n)$ is bounded, say $|a_n \rho^n| \leq K$ for all n. Then

$$|a_n||x-c|^n = |a_n| \left(\frac{|x-c|}{\rho}\right)^n \rho^n = \left(\frac{|x-c|}{\rho}\right)^n |a_n|\rho^n \leq K \left(\frac{|x-c|}{\rho}\right)^n$$

and the geometric series $\sum \left(\frac{|x-c|}{\rho}\right)^n$ converges as $\frac{|x-c|}{\rho} < 1$. So by comparison, $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely for such x, and therefore converges for all x. □

Lemma 1.22. If $R = 0$, then the power series converges only at $x = c$ (trivial), and if $R = \infty$ then the power series converges for all $x \in \mathbb{R}$.

Lemma 1.23.
i) if $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is equal to R.

ii) if $\lim_{n\to\infty} |a_n|^{-1/n}$ exists, then it is equal to R.

Theorem 1.24. Assume $R > 0$. Suppose that $0 < r < R$. Then the power series with radius R converges uniformly and absolutely on $|x - c| \leq r$ to a continuous function f:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

defines a continuous function $f : (c - R, c + R) \to \mathbb{R}$.

Proof. We have already seen the absolute convergence. With the same notation as before, we have that $(for r < \rho < R)$:

$$|a_n||x-c|^n \leq K\left(\frac{r}{\rho}\right)^n := M_n$$

for all x with $|x - c| \leq r$. Since $\sum M_n$ converges, by the Weierstrass M-test the sum converges uniformly on $[c-r, c+r]$. Since each $a_n(x-c)^n$ is a continuous function, so is the limiting function $f : [c-r, c+r] \to \mathbb{R}$. Since $r < R$ is arbitrary, f is defined and continuous on $(c - R, c + R)$. □

Lemma 1.25. The two power series $\sum_{n=1}^{\infty} a_n(x-c)^n$ and $\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ have the same R.

Proof. Let the radii of convergence be R_1 and R_2 respectively. Since $|a_n r^n| \leq |na_n r^n|$ for $n \geq 1$, $R_2 \leq R_1$. Now suppose for contradiction's sake that $R_2 < R_1$. Then we can choose a ρ and r such that $R_2 < \rho < r < R_1$, and such that $(a_n r^n)$ is bounded, say $|a_n r^n| \leq K$ for all n. Then

$$|na_n\rho^n| = |a_n r^n| \times n(\rho/r)^n \leq K \times n(\rho/r)^n.$$

The sequence $n(\rho/r)^n$ converges to 0 as $\rho < r$, and therefore $(na_n\rho^n)$ is also bounded. This contradicts the definition of R_2 and so $R_1 = R_2$. □

Theorem 1.26. Suppose the radius of convergence of a power series is R. Then

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

is infinitely differentiable on $|x - c| < R$, and for such x,

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$$

and the series converges absolutely and uniformly on $[c - r, c + r]$ for any $r < R$. Moreover, $a_n = \frac{f^{(n)}}{n!}$.

Proof. Consider the series $\sum_{n=0}^{\infty} na_n(x-c)^{n-1}$ which has radius of convergence R and so converges uniformly on $[c - r, c + r]$ for any $r < R$. Since $na_n(x - c)^{n-1}$ is the derivative of $a_n(x - c)^n$, and since the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at at least one point, $f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$. The term follows by repeated differentiation. □

The Exponential Function

The power series given by $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x, i.e. has $R = \infty$. Let

$$E(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Then E converges absolutely for all x and uniformly on any closed, bounded interval $[-r, r]$ to the infinitely differentiable function that satisfies:

$$E'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Proposition 1.27. The power series $E(x)$ satisfies the following properties:

- $E'(x) = E(x)$ for all x
- $E(x)E(-x) = 1$ for all x
- $E(x) > 0$ for all x
- $E(x+y) = E(x)E(y)$
- $E(-x) = E(x)^{-1}$
- $E(kx) = E(x)^k$ for $k \in \mathbb{N}, \mathbb{Q}$
- $E(x/k) = E(x)^{1/k}$ for $k \in \mathbb{N}$
- $E(q) = E(1)^q$ for all $q \in \mathbb{Q}$

Definition 1.28 (Analytic Function)**.** We say that a function is analytic when it possesses a power series expansion. That is, f is analytic on $\{x \mid |x-c| < r\}$ if there is a power series which converges to f on $\{x \mid |x-c| < r\}$. Analytic functions are infinitely differentiable on $\{x \mid |x-c| < r\}$, and the coefficients of the power series are uniquely determined by $a_n = \frac{f^{(n)}(c)}{n!}$.

A function is analytic on $\{x \mid |x-c| < r\}$ iff

$$f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(c)(x-c)^j}{j!} \to 0$$

as $n \to \infty$ for all $x, |x-c| < r$.

Example 1.29. Prove that the function $f(x) = e^{\frac{-1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$ does not admit a power series expansion (is not analytic).

Proof. For a function to admit a power series expansion, it must be analytic. Consider that $\forall j$, f is j times differentiable and there exist polynomials q_j such that $f^{(j)}(x) = e^{\frac{-1}{x^2}} q_j(1/x)$ for $x \neq 0$ and $f^{(j)}(0) = 0$. Consider

$$R_{n+1}f(x,0) = \frac{f^{(n+1)}(\xi)x^{n+1}}{(n+1)!}$$

But $f^{(n+1)}(\xi) \not\to 0$ when $x \neq 0$ by the above, and so the only time $R_{n+1}f(x,0) \to 0$ is when $x = 0$. So, $R_{n+1}f(x,0) \not\to 0$ as $n \to \infty$ for all $x \in (-r,r)$, and so f is not analytic. □

2 Integration of Functions $f : \mathbb{R} \to \mathbb{R}$

Riemann Integration

Definition 2.1 (Characteristic Functions)**.** If $E \subseteq \mathbb{R}$, we define its **characteristic function** $\chi_E : \mathbb{R} \to \mathbb{R}$ by

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Let I be a bounded interval. Then $\int \chi_I = \text{length}(I)$.

Definition 2.2 (Step Functions)**.** We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a step function if there exist real numbers $x_0 < x_1 < ... < x_n$ for some $n \in \mathbb{N}$ such that

1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$.
2. $\phi(x)$ is constant on (x_{j-1}, x_j) for $1 \leq j \leq n$.

We can write $\phi(x) = \sum\limits_{j=1}^{n} c_j \chi_{(x_{j-1},x_j)}(x)$.

Proposition 2.3. The class of step functions forms a vector space.

Proof. Let ϕ and ψ be step functions and a and b real numbers. Then we can write $\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1},x_j)}(x)$ with respect to points $X = \{x_0, ..., x_n\}$ and $\psi(x) = \sum_{j=1}^{n} d_j \chi_{(x_{j-1},x_j)}(x)$ with respect to points $Y = \{y_0, ...y_\}$.

Then consider that both ϕ and ψ are step functions with respect to $Z = X \cup Y$. Let $m = n + k$. Then

1. Definitiely $\phi + \psi$ is zero outside z_0 and z_m, as both functions are separately zero outside of these two values.
2. $\phi + \psi$ is constant on (z_{j-1}, z_j) for all j: this is true as the interval $(z_{j-1}, z_j) \subset (x_{j-1}, x_j)$ and $(z_{j-1}, z_j) \subset (y_{j-1}, y_j)$, where the functions are constant. The sum of two constants is constant.

□

Clearly $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$ are also step functions, and so is $\phi\psi$.

Proposition 2.4. If ϕ and ψ are step functions and $a, b \in \mathbb{R}$, then

$$\int a\phi + b\psi = a \int \phi + b \int \psi.$$

Proposition 2.5. If ϕ and ψ are step functions and $\phi \geq \psi$, then

$$\int \phi \geq \int \psi.$$

Definition 2.6. Let $f : \mathbb{R} \to \mathbb{R}$. We say a function f is Riemann integrable if for every $\epsilon > 0$ there exist step functions ϕ and ψ such that $\phi \leq f \leq \psi$ and $\int \psi - \int \phi < \epsilon$

Proposition 2.7. If f is Riemann integrable, then f is bounded and has bounded support.

3 Metric Topology of Euclidean Spaces

Definition 3.1. (Simmons) Let X be a **non-empty set**. Then a metric on X is a real function d of ordered pairs of elements in X, $d : X \times X \to \mathbb{R}_+$, such that the following conditions hold:

1. $d(x, y) \geq 0$ (the codomain of d is non-negative)
2. $d(x, y) = d(y, x)$ (d is symmetric)
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality holds for d), for all $x, y, z \in X$.

We define a metric space to be a set X with a metric d on X. We denote the metric space by (X, d).

3.1 Examples

Example 3.2. A common metric used in $\mathbb{R}$ is the following:

$$d(x, y) = |x - y| \tag{1}$$

where, of course, $|x - y|$ is the absolute value of $x - y$. That is,

$$|x - y| = \max\{(x - y), -(x - y)\}. \tag{2}$$

We will prove that this is a metric in $\mathbb{R}$:

Proof.

1. Of course, $|x - y| \geq 0 \ \ \forall\ x, y \ \in \mathbb{R}$, this is trivial.
2. Note that $|x - y| = \max\{(x - y), -(x - y)\} = \max\{-(y - x), (y - x)\} = |y - x|$.
3. ($\Rightarrow$) If $|x - y| = 0$, then either $x - y = 0$, or $y - x = 0$. In either case, $x = y$. ($\Leftarrow$) If $x = y$, then $x - y = 0$ so $|x - y| = 0$.
4. We wish to show that $|x - y| \leq |x - z| + |z - y|$:
   Consider that

$$\begin{aligned}
(|x + y|)^2 =& |x + y||x + y| = |(x + y)(x + y)| \\
\Rightarrow (|x + y|)^2 =& |x^2 + 2xy + y^2| \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \\
\Rightarrow (|x + y|)^2 \leq& (|x| + |y|)^2
\end{aligned}$$

   and as $|x + y| \geq 0$ and $|x| + |y| \geq 0 \ \forall x, y, \in \mathbb{R}$,

$$\Rightarrow |x + y| \leq |x| + |y|.$$

   This is a version of the triangle inequality. We have that $|x - y| = |x - z + z - y|$, so, applying the above, $|x - y| \leq |x - z| + |z - y|$, as required. Hence, $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

□

Example 3.3. Let X be any arbitrary set. We define the discrete metric on X to be:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

To prove that this is a metric, consider the following:

Proof. 1. Clearly, $d(x, y) \geq 0$ for all x, y in X.

2. If $x = y$ then $y = x$, and similarly, if $x \neq y$ then $y \neq x$. Hence, $d(x, y) = d(y, x)$. This is trivial.

3. By definition $x = y \iff d(x, y) = 0$.

4. Consider that $d(x, y)$, $d(x, z)$, and $d(z, y)$ are all either 0 or 1. So, it is easy to see that regardless, the triangle inequality will hold. The only possible place where the triangle inequality may not hold is when $d(x, y) = 1$ and $d(x, z) = d(z, y) = 0$. Suppose that it does not hold. By definition $x = z$ and $z = y$ so $x = y$, which is a contradiction. The rest of the cases are trivial, and so the triangle inequality holds for all values x, y, z.

□

Example 3.4. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space with elements $x = (x_1, ..., x_n)$, $(x_i \in \mathbb{R})$, and let

$$|x| = \sqrt{\sum_{i=1}^{n} {x_i}^2} \geq 0$$

The usual or Euclidean metric defined on the space $\mathbb{R}^n$ is given by:

$$d(x, y) = |x - y|.$$

To prove that this is a metric, consider the following:

Proof. 1. By definition, $d(x, y) \geq 0$.

2. $|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = |y - x|$.

3. Suppose $x = y$. Then $|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (x_i - x_i)^2} = 0$. Conversely, suppose $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$. As squares are always non-negative, $x_i = y_i$ for all $i = 1, ..., n$ and so $x = y$.

4. Take any $x, y, z \in \mathbb{R}^n$. Let $x_i - y_i = r_i$, $y_i - z_i = s_i$, and all summations be over $i = 1, ..., n$. Then we wish to prove that:

$$\left(\sum (r_i + s_i)^2\right)^{1/2} \leq \left(\sum r_i^2\right)^{1/2} + \left(\sum s_i^2\right)^{1/2}.$$

As both sides are non-negative, this is equivalent to proving the square of both sides, that is:

$$\sum r_i^2 + \sum s_i^2 + 2\sum r_i s_i \leq \sum r_i^2 + \sum s_i^2 + 2\left(\sum r_i^2\right)^{1/2}\left(\sum s_i^2\right)^{1/2}.$$

Simplifying, this is just the Cauchy-Schwarz Inequality:

$$\sum_{i=1}^{n} (r_i s_i) \leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} \tag{3}$$

Cauchy-Schwarz Inequality Proof:
We will prove 3 by induction. Let $P(n)$ be the above statement. $P(1)$ obvious, as

$$\sqrt{\left(\sum_{i=1}^{1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{1} s_i^2\right)} = r_i s_i = \sum_{i=1}^{1} (r_i s_i).$$

To prove $P(2)$, observe that

$$
\begin{aligned}
& r_1s_1 + r_2s_2 \leq \sqrt{r_1^2 + r_2^2}\sqrt{s_1^2 + s_2^2} \\
\iff & (r_1s_1 + r_2s_2)^2 \leq (r_1^2 + r_2^2)(s_1^2 + s_2^2) \\
\iff & r_1^2s_1^2 + 2r_1s_1r_2s_2 + r_2^2s_2^2 \leq r_1^2s_1^2 + r_1^2s_2^2 + r_2^2s_1^2 + r_2^2s_2^2 \\
\iff & 0 \leq r_1^2s_2^2 - 2r_1s_1r_2s_2 + r_2^2s_1^2 \\
\iff & 0 \leq (r_1s_2 - r_2s_1)^2
\end{aligned}
$$

Which is true. Hence $P(2)$ holds.

Assume that $P(n)$ is true. Then,

$$
\begin{aligned}
\sum_{i=1}^{n}(r_is_i) &\leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} \\
\Rightarrow \sum_{i=1}^{n}(r_is_i) + (r_{n+1}s_{n+1}) = \sum_{i=1}^{n+1}(r_is_i) &\leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1}s_{n+1})
\end{aligned}
$$

But by $P(2)$,

$$
\begin{aligned}
& \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1}s_{n+1}) \leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right) + r_{n+1}^2}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right) + s_{n+1}^2} \\
\Rightarrow & \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1}s_{n+1}) \leq \sqrt{\left(\sum_{i=1}^{n+1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n+1} s_i^2\right)}
\end{aligned}
$$

Hence, we have that

$$
\sum_{i=1}^{n+1}(r_is_i) \leq \sqrt{\left(\sum_{i=1}^{n+1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n+1} s_i^2\right)}
$$

And so $P(n+1)$ holds whenever $P(n)$ holds .

Thus, the triangle inequality holds for all $x, y, z \in \mathbb{R}^n$.

□

3.2 Open and Closed Sets in Metric Spaces

3.2.1 Open Sets

We first define an open ball, and hence an open set:

Definition 3.5. Let (X, d) be a metric space. If x_0 is a point of (X, d) and r is a positive real number, then the open ball $B_r(x_0)$ with centre x_0 and radius r is the subset of X defined by

$$
B_r(x_0) := \{x \in X \mid d(x, x_0) < r\} \tag{4}
$$

A subset G of the metric space (X, d) is called an open set if and only if, given any point x in G, there exists a positive real number r such that $B_r(x) \subseteq G$.

Proposition 3.6. In any metric space X, each open ball is an open set.

Proof. Let $B_r(x_0)$ be an open ball in X. Take any x in $B_r(x_0)$. We wish to prove that there exists an r_1 such that $B_{r_1}(x) \subseteq B_r(x_0)$. Consider that, for any $x \in B_r(x_0)$, $d(x, x_0) < r$. So, take $r_1 = r - d(x, x_0)$, a positive real number. Then, for any $y \in B_{r_1}(x)$, $d(y, x) < r_1$, and by the triangle inequality $d(y, x_0) \leq d(y, x) + d(x, x_0) < r_1 + d(x, x_0) = (r - d(x, x_0)) + d(x, x_0) = r$. That is, $d(y, x_0) < r$, and so $y \in B_r(x_0)$. Hence, as y is arbitrary, $B_{r_1}(x) \subseteq B_r(x_0)$, and thus the open ball $B_r(x_0)$ is an open set. □

There are several important properties of open sets with respect to metric spaces:

Proposition 3.7. Let (X, d) be a metric space. Then the empty set $\emptyset$ and the full space X are open sets.

Proof. Let us first prove that the empty set $\emptyset$ is open. Note that $\emptyset \subset X$. We can call $\emptyset$ open if we can show that for any point in $\emptyset$, there exists a positive real number r such that $B_r(x) \subseteq \emptyset$. But of course, there are no points in the empty set, so this condition automatically holds. To prove that X, the full space, is open, consider that the full space contains every possible open ball in X. Hence, for all x in X, there exists an open ball $B_r(x) \subseteq X$. Hence X is open. □

Proposition 3.8. The intersection of any finite collection of open sets in a metric space X is open in X.

Proof. Let U and V be open sets in a metric space X. Then by definition, there exist positive real numbers r_1 and r_2 such that $B_{r_1}(x) \subseteq U$ and $B_{r_2}(x) \subseteq V$ for all $u \in U$, $v \in V$. Now take any $x \in U \cap V$. Then we have that there exist positive real numbers r_1 and r_2 such that $B_{r_1}(x) \subseteq U \cap V$ and $S_{r_2}(x) \subseteq U \cap V$. Take $r = \min\{r_1, r_2\}$. Then we have that $B_r(x) \subseteq U \cap V$ (It may be useful to remember that both of these open balls are centred on x). Hence, the intersection $U \cap V$ of two open sets U,V, is also open. It follows by induction that the intersection of a finite number of open sets is also open. □

Proposition 3.9. The union of any arbitrary collection of open sets in a metric space X is also open in X.

Proof. Let $x \in \cup_{i \in I} U_i$, with $\{U_i\}$ a (possibly infinite) collection of open sets. Then x is an interior point (refer to Definition 3.13) of some U_k and there is an open ball centred on x contained in U_k by definition of open sets. This ball is therefore contained in $\cup_{i \in I} U_i$, and as x is arbitrary the union is open. Note that this proof does not rely on the assumption that the union is finite. □

Remark 3.10. Note that Proposition 3.8 only applies to **finite** intersections, not **infinite** intersections. The following counterexample demonstrates this.

Example 3.11. Take the metric space $\mathbb{R}$ under the usual metric. Consider the open sets given by $(-\frac{1}{n}, \frac{1}{n})$. The infinite intersection of these intervals is the singleton $\{0\}$, which is not open. If it were true that singleton sets were open, then as every set can be written as an arbitrary union of singleton sets, by Proposition 3.9 the set $[0, 1)$ would be open. It is, of course, not open, as there is no $r > 0$ such that $B_r(0) \subseteq [0, 1)$.

The following is quite useful to know:

Theorem 3.12. Let X be a metric space. A subset G of X is open $\iff$ it is a union of open balls.

Proof. Let X be a metric space. We will first prove the forwards implication and then the backwards implication:
$\Rightarrow$: Suppose we have a subset G of X, and that G is open. Then, by definition, for every point x in G, there exists an open ball centred on x of radius r such that $x \in B_r(x) \subseteq G$. Then, as for each x in G there is an open ball containing x, then we can write G as the union of all the open balls $B_r(x)$ for each x. That is, if G is open, then G is a union of open balls.
$\Leftarrow$: We already know by Proposition 3.6 that an open ball is an open set, and so by Proposition 3.9, the union of open balls is open. So, if G is a union of open balls, then G is open. □

A fairly useful concept is the **interior** of a subset U:

Definition 3.13. Let X be a metric space and $U \subseteq X$. A point $x \in U$ is said to be an **interior** point of U if there exists a positive real number $r > 0$ such that the ball centred at x with radius r is a subset of U, that is, $B_r(x) \subseteq U$. The interior of U refers to the set of all interior points of U, and is denoted $\text{int}(U)$.

Remark 3.14. The interior of a subset, $\text{int}(U)$, is always open.

Proof. We have that for each $x \in \text{int}(U)$, there exists an $r > 0$ such that $B_r(x) \subseteq \text{int}(U)$. So we can write the interior of U as the union of open balls. By Theorem 3.12, it follows that $\text{int}(U)$ is open. □

Example 3.15. Equipped with the usual metric, the following examples help to visualise open balls in certain metric spaces:

1. In $\mathbb{R}$, $B_r(x_0) = (x_0 - r, x_0 + r)$.
2. In $\mathbb{R}^2$, $B_r(x_0)$ is the interior (refer to Definition 3.13) of a disc of radius r.
3. In $\mathbb{R}^3$, $B_r(x_0)$ is the interior (refer to Definition 3.13) of a sphere of radius r.

Note that $B_r(x_0)$ depends in general on the metric, d, as well as the underlying set.

Definition 3.16. Let X be a metric space and $U \subseteq X$. A point $x \in U$ is said to be a boundary point of U if for every positive real number $r > 0$ we have that there exists points $a, b \in B_r(x)$ such that $a \in U$ and $b \in U'$. The boundary of U refers to the set of all boundary points of U, and is denoted ∂U.

3.2.2 Closed Sets

Definition 3.17. Let X be a metric space. A subset A of X is closed $\iff$ its complement A' is open.

This is equivalent to the following definition:

Definition 3.18. A subset A of the metric space X is called a closed set if it contains each of its limit points.

where limit points are defined to be the following:

Definition 3.19. If A is a subset of X, a point x in X is called a limit point of A if each open centred on x contains at least one point of A different from x.

To prove that these two definitions of closed sets are equivalent, consider the following:

Proof. Suppose that A is closed. Then A' is open. Also suppose x is a limit point of A. Then if $x \notin A$, then there is some open set U such that $x \in U \subset A'$, which contradicts x being a limit point. So A contains all of its limit points.

Now suppose that A contains all of its limit points, and take some $x \notin A$. Since x cannot be a limit point, there is some open set $U \in A'$ such that $x \in U$. So, A' is open and hence A is closed. □

The following are several important properties of closed sets:

Proposition 3.20. In any metric space X, the empty set $\emptyset$ and the full space X are closed sets.

Proof. The empty set has no elements, and so really contains all of its limit points, and is therefore closed. To prove that the full space X is closed, consider that it contains all points, and so automatically contains all of its limit points and hence is closed. □

Definition 3.21. Let (X, d) be a metric space. If x_0 is a point of (X, d) and r is a positive real number, then the closed ball $D_r(x_0)$ with centre x_0 and radius r is the subset of X defined by

$$D_r(x_0) := \{x \in X \mid d(x, x_0) \leq r\} \tag{5}$$

Proposition 3.22. In any metric space X, each closed ball is a closed set.

Proof. Let X be a metric space with metric d. Consider an arbitrary closed ball $D = D_r(x_0)$ centred on x_0 with radius r in X. The claim is equivalent to showing that D' is open, by Definition 3.17. That is, we need to show that for every $y \in D'$, there exists an open ball centred on y contained in D'. Since $y \notin D$, then $d(x_0, y) > r$. So, $d(x_0, y) - r > 0$. Define $r_1 = d(x_0, y) - r$. We claim that the open ball $B_{r_1}(y)$ is contained in D'. Consider any z in $B_{r_1}(y)$. Then by the triangle inequality,

$$\begin{aligned}
&d(x_0, y) \leq d(x_0, z) + d(z, y)\\
\Rightarrow &d(x_0, z) \geq d(x_0, y) - d(z, y) > d(x_0, y) - r_1\\
\Rightarrow &d(x_0, z) > d(x_0, y) - (d(x_0, y) - r) = r\\
\Rightarrow &d(x_0, z) > r.
\end{aligned}$$

Hence, z is not contained in D. As z and y are arbitrary points in D', it follows that D' is open, and so D is closed. That is, the closed ball $D_r(x_0)$ is a closed set. □

Proposition 3.23. Let X be a metric space. Then

1. Any arbitrary intersection of any collection of closed sets in X is closed.
2. The finite union of any collection of closed sets in X is closed.

Proof. To prove the above, we can use De Morgan's Law:

Lemma 3.24. De Morgan's Law: Let S and T be sets, and let $\{T_i\}_{i \in I}$ be a collection of subsets of T. Then

$$S \setminus \bigcap_{i \in I} T_i = \bigcup_{i \in I} (S \setminus T_i)$$

Both of these follow directly from De Morgan's Law, Proposition 3.8, and Proposition 3.9. □

Example 3.25. Every subset of the discrete space X is both open and closed.

Proof. Take any arbitrary $U \subseteq \mathbb{R}$. If $U = \emptyset$ or $U = \mathbb{R}$, then we are done as we have already proven that the empty set and the full space are both open and closed for any metric space. Take any $x \in U$. Consider that at the most extreme, $U = \{x\}$ and so consider that the ball $B_1(x) = \{x\}$, and hence we have found a ball contained in U centred at x. By definition, U is open. By Definition 3.17 the complement U' is closed. And as U is arbitrary, we have that all subsets of the discrete space $\mathbb{R}$ are both open and closed. □

3.3 Continuity in Metric Spaces

We will begin with the most fundamental and intuitive definition of continuity:

Definition 3.26. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ continuous iff for every $a \in X$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow \rho\left(f(x), f(a)\right) < \epsilon. \tag{6}$$

This is equivalent to the following definition:

Definition 3.27. A map $f : M_1 \to M_2$ between metric spaces is continuous at x in M_1 if given any $S_\epsilon(f(x))$ there exists a $B_\delta(x)$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Proposition 3.28. Suppose that $f : M_1 \to M_2$ is a map between metric spaces. Then f is continuous $\iff$ for every set G open in M_2, $f^{-1}(G)$ is open in M_1.

Proof. We will proceed by proving the forwards implication, followed by the backwards implication of the above statement:

$\Longrightarrow$: Suppose $f : M_1 \to M_2$ is continuous, and suppose we have an open set G in M_2. We wish to prove that $f^{-1}(G)$ is open in M_1. Take any $x \in f^{-1}(G)$. Then $f(x) \in G$. Hence, by definition of an open set, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset G$. And so, by definition of continuity, there exists a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$. So, as $B_\epsilon(f(x)) \subset G$, then $f(B_\delta(x)) \subset G$ and $B_\delta(x) \subset f^{-1}(G)$. Hence, $f^{-1}(G)$ is open in M_1.

$\Longrightarrow$: Now suppose that we have that, for every open set G in M_2, $f^{-1}(G)$ is open in M_1. Consider an arbitrary point x in M_1, then $f(x) \in M_2$. By definition of an open set, there exists an $\epsilon > 0$ such that $B_\epsilon(f(x))$ is open in M_2. And so by assumption, $f^{-1}(B_\epsilon(f(x)))$ is open in M_1. But we know that $f(x) \in B_\epsilon(f(x))$ and so $x \in f^{-1}(B_\epsilon(f(x)))$. By definition of an open set, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. But this means that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, which is the definition of continuity. That is, f is a continuous function.

Hence, f is continuous $\iff$ for every set G open in M_2, $f^{-1}(G)$ is open in M_1.

□

3.4 Equivalent Metrics

Definition 3.29. We say that two metrics d_1 and d_2 on a set X are equivalent if the identity map $i : (X, d_1) \to (X, d_2)$ is continuous, and if the map $i^{-1} : (X, d_2) \to (X, d_1)$ is continuous.

3.5 Convergence In a Metric Space

Definition 3.30. A sequence (x_n) of points in a metric space X with metric d **converges** to a point x in X if given any (real number) $\epsilon > 0$, there exists (an integer) N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$.

This can equivalently be written as:

Definition 3.31. A sequence (x_n) of points in a metric space X with metric d converges to a point x (that is, $\lim_{n\to\infty}(x_n) = x$) if, given any $\epsilon > 0$, there exists an integer N such that

$$n \geq N \Rightarrow d(x, x_n) < \epsilon.$$

Theorem 3.32. Let X be a metric space, and let $U \subseteq X$. If $x \in X$ is a limit point of U, then there exists a sequence $(x_n) \in U$ such that $\lim_{n\to\infty} x_n = x$.

Proof. Take any $x_0 \in U, x_0 \neq x$. Then take some $x_1 \in U \cap B_{\frac{d(x,x_0)}{2}}, x_1 \neq x$. We continue this process and define x_n to be some point in $U \cap B_{\frac{d(x,x_n)}{2}}, x_n \neq x$. The existence of this point is guaranteed by the definition of a limit point. Clearly (x_n) is a convergent sequence tending to x. □

Theorem 3.33. Let X be a metric space and $U \subseteq X$. Then U is closed iff the limit of every convergent sequence $(x_n) \in U$ satisfies

$$\lim_{n\to\infty} x_n \in U.$$

Proof. We will first prove the forwards implication and then the backwards implication:
$\Rightarrow$: Suppose that $U \neq \emptyset$ is closed, and suppose there exists a convergent sequence (x_n) in U whose limit converges to a point x not in U (i.e. $x \in U'$). By definition of convergence, given any $\epsilon > 0$, there exists some N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. We know that as U is closed, by Definition 3.17 U' is open. By definition of openness, there is some $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq U'$. But then there is some N such that for all $n \geq N$, $x_n \in B_{\epsilon_0}(x)$, which would imply that some terms of the sequence (x_n) are outside of U, which is a contradiction.
$\Leftarrow$: Now suppose that the limit of every convergent sequence $(x_n) \in U$ satisfies $\lim_{n\to\infty} x_n \in U$. Then by Theorem 3.32, all limit points of U are contained within U. So, by Definition 3.18, U is closed. □

Definition 3.34. Cauchy: A sequence (x_n) in a metric space X is Cauchy if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $d(x_n, x_m) < \epsilon$.

Definition 3.35. Bounded: A sequence (x_n) in a metric space X is bounded if there is an $M > 0$ and a $b \in X$ such that $d(x_n, b) \leq M$ for all $n \in \mathbb{N}$.

Proposition 3.36. Let X be a metric space.

1. A sequence in X can have at most one limit.
2. If $x_n \in X$ converges to A and (x_{n_k}) is any subsequence of (x_n), then x_{n_k} converges to a as $k \to \infty$.
3. Every convergent sequence in X is bounded.
4. Every convergent sequence in X is Cauchy.

Proposition 3.37. Let X be a metric space and let (x_n) be a Cauchy sequence. Then (x_n) converges to x iff (x_n) has a subsequence that converges to x.

Proof. $\Rightarrow$: This trivially follows as (x_n) is a subsequence of itself that converges to x.
$\Leftarrow$: Suppose (x_n) is a Cauchy sequence with a subsequence (x_{n_k}) that converges to x. By definition of a Cauchy sequence, given any $\epsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that $n, m \geq N_1$ implies that $d(x_n, x_m) < \epsilon/2$. And as (x_{n_k}) is a convergent subsequence, then by definition of convergence, there exists an $N_2 \in \mathbb{N}$ such that $d(x_{n_k}, x) < \epsilon/2$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, by the triangle inequality, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon$. So, for all $n \geq N$, $d(x_n, x) < \epsilon$ and so (x_n) converges to x. □

3.5.1 Uniform Convergence in Metric Spaces

The following propositions will be of use later when discussing Function Spaces.

Definition 3.38. We say that a subset $U \subset X$ of a metric space X is bounded if $U \subset B_r(x)$ for some $r > 0$ and $x \in X$.

Definition 3.39. We say that a **function** $f : X_1 \to X_2$ is **bounded** if $f(X_1) \subset X_2$ is bounded.

Definition 3.40. We say that a sequence (f_n) of functions $f : X_1 \to X_2$ is **uniformly convergent** to a function $f : X_1 \to X_2$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d_2(f_m(x), f_n(x)) < \epsilon$ for all $x \in X_1$.

It is important to note the difference between pointwise convergence and uniform convergence. Uniform convergence is a stronger notion. If a sequence converges uniformly, it is guaranteed to converge under the given metric. It is possible for a sequence to converge pointwise to a point, but not converge with respect to the particular metric. The following example demonstrates this fact.

Example 3.41. Consider the sequence (f_n), $f_n(x) = \frac{x}{n}$ in the metric space $\mathbb{R}$ with the usual metric. The sequence is pointwise convergent to 0:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0$$

But (f_n) is not uniformly convergent. Take $\epsilon = 1$ for example. Supposing (f_n) is uniformly convergent, then there is some integer N such that $m, n > N$ implies that $|\frac{x}{m} - \frac{x}{n}| < 1$. But we can take x to be arbitrarily large and $m \neq n$, so this is a contradiction.

Proposition 3.42. Let (f_n) be a sequence of functions $f_n : X_1 \to X_2$. If each f_n is bounded and $f_n \to f$ uniformly, then $f : X_1 \to X_2$ is bounded.

Proof. Denote the metrics associated with X_1 and X_2 by d_1 and d_2 respectively. Supposing f_n is uniformly convergent, by Definition 3.40 there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d_2(f_m(x), f_n(x)) < 1$ for all $x \in X_1$. And supposing f_n is bounded, by Definition 3.39, there exists some $f(x_0) \in X_2$ and some $r > 0$ such that $d_2(f_n(x), f(x_0)) < r$ for all $x \in X_1$. By the triangle inequality,

$$d_2(f(x), f(x_0)) \leq d_2(f(x), f_n(x)) + d_2(f_n(x), f(x_0)) < 1 + r.$$

Hence, by definition f is bounded. □

Definition 3.43. We say that a sequence of functions (f_n), $f_n : X_1 \to X_2$, is uniformly Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d(f_m(x), f_n(x)) < \epsilon$ for all $x \in X_1$.

4 Completeness and Contraction Mappings

Definition 4.1. We say that a metric space X is **complete** iff every Cauchy sequence $(x_n) \in X$ converges to some point in X.

It will be useful to recall the definition of a Cauchy sequence in a metric space, Definition 3.34.

Remark 4.2. $\mathbb{R}$ equipped with the usual metric is a complete metric space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ is bounded, and so all are all subsequences of $\{x_n\}$. There exists a subsequence of $\{x_n\}$ that is monotone and bounded, and hence convergent. But if a Cauchy sequence has a subsequence that converges to some x in $\mathbb{R}$, then it too converges to x. Hence, $\{x_n\}$ is convergent to some x in $\mathbb{R}$. □

Theorem 4.3. Let X be a complete metric space and U a subset of X. Then U (as a subspace) is complete iff U (as a subset) is closed.

Proof. $\Rightarrow$: Suppose that U is complete and take some sequence $(x_n) \in U$ that converges. Any convergent sequence is a Cauchy sequence, and so (x_n) is Cauchy. By assumption, if x is the limit of (x_n), then $x \in U$. By Theorem 3.33, it follows that U is closed.
$\Leftarrow$: Now suppose that U is a closed subset and that (x_n) is Cauchy in U. Then (x_n) is also Cauchy in X as U is a subspace of X. So (x_n) converges to some x in X. But by assumption, U is closed and so by Theorem 3.33, it follows that x must be in U, and so U is complete. □

5 Compactness in Metric Spaces

5.1 Basic Definition

A cover for a set X is a collection $\mathcal{C}$ of subsets of X such that $X \subset \bigcup_{\lambda \in \Lambda} C_\lambda$. In the context of topological spaces, we can write this definition formally as:

Definition 5.1. If (X, d) is a metric space and $A \subseteq X$, then a collection of subsets $\mathcal{C}$ of X is said to be a cover of A if

$$A \subseteq \bigcup_{\lambda \in \Lambda} C_\lambda$$

Definition 5.2. A subcover $\mathcal{V}$ of a given cover $\mathcal{C}$ for a set X is a subcollection $\mathcal{V} \subset \mathcal{C}$ which still forms a cover for X.

Definition 5.3. An **open** cover of a metric space $\{X, \tau\}$ is a collection $\{C_\lambda \mid \lambda \in \Lambda\}$ of **open** subsets C_λ of X such that

$$\bigcup_{\lambda \in \Lambda} C_\lambda = X.$$

Remark 5.4. If a cover $\mathcal{C}$ is open, then a subcover $\mathcal{V} \subset \mathcal{C}$ is also open. This is trivial.

We say that a cover is finite if $\mathcal{C}$ is finite.

Definition 5.5. A compact space is a metric space in which every open cover has a finite subcover. Symbolically, whenever $\mathcal{C} = \{C_\lambda \mid \lambda \in \Lambda\}$ is an open cover, there exist $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$X = \bigcup_{j=1}^{n} C_{\lambda_j}.$$

It is important to understand this definition well. In layman's terms, given any open cover $\mathcal{C}$ of X, there exists a finite number of the open sets in $\mathcal{C}$ which are enough to cover X. The key concept is that **every** open cover is finite, not just **(at least) one** open cover is finite. The latter is trivially true, taking the singleton collection $\{X\}$ as a finite open cover.

Definition 5.6. Let X be a metric space. Then a subspace $A \subseteq X$ is said to be compact in X iff A itself is a compact metric space.

Proposition 5.7. Let (X, τ) be a metric space and $A \subseteq X$. Then A is compact iff every open cover of A has a finite subcover.

Proof. We will first prove the forwards implication and then the backwards implication.
($\Rightarrow$) Suppose that X is a metric space, A is a compact subset of X, and that $\{C_\lambda\}$ is an open cover for X. By definition of compactness, there exists a finite subset $\Lambda' \subset \Lambda$ such that $\{C_\lambda\}_{\lambda \in \Lambda'}$ is a finite open cover for X. Then $\{C_\lambda \cap A\}_{\lambda \in \Lambda'}$ is a finite collection of open sets whose union covers A. Thus, as the original choice of open cover is arbitrary, we can construct finite subcovers for any open cover of a subset A and so the forwards implication holds.
($\Leftarrow$) Suppose we have a metric space X and that every open cover of a subset A of X has a finite subcover. Take an arbitrary collection of open sets in A, $\{U_\lambda\}$, such that the union of the open sets equals A. Each U_λ can be written in the form $U_\lambda = V_\lambda \cap A$, where V_λ is an open set in X. It follows that $U_\lambda \subset V_\lambda$ and so $\{V_\lambda\}$ forms an open cover for A in X. By assumption each open cover of A has a finite subcover, and so $\{V_\lambda\}$ has a finite subcover. Thus A is compact and the backwards implication holds.

□

5.2 Properties of and Theorems Relating to Compact Spaces

Several important theorems and propositions follow from these definitions:

Theorem 5.8. Let X be a metric space, and $A \subseteq X$ be a finite subspace. Then A is compact in X.

Proof. Suppose A is a finite subset of X. Then we can write $A = \{x_1, x_2, ..., x_n\}$. Let $\mathcal{C} = \{C_i \mid i \in I\}$ be an open cover of A. Then by Definition 5.1,

$$A \subseteq \bigcup_{i \in I} C_i.$$

At most, A can be partitioned into n groups (as there are n elements), where all elements are contained in $\mathcal{C}$. So there exists a subcollection $I^* \subseteq I$ such that

$$A \subseteq \bigcup_{i \in I^*} C_i$$

(where $|I^*| \leq |I|$). By definition, $\mathcal{C}^*$, the subset relating to I^*, defines a subcover of A. The choice of $\mathcal{C}$ is arbitrary. So A is compact in X. □

Remark 5.9. From the above theorem, it follows that if (X, τ) is a metric space, and X is finite, then (X, τ) is a compact metric space.

Proposition 5.10. Let $f : X \to Y$ be a continuous map between metric spaces. If X is compact, so is $f(X)$.

Proof. Let U_λ be open subsets of Y which cover $f(X)$. Then $f^{-1}(U_\lambda)$ are open sets in X which cover X. Hence, as X is compact, there exists a finite subcover $\{f^{-1}(U_{\lambda_1}), ...f^{-1}(U_{\lambda_n})\}$ in this cover of X and so $\{U_{\lambda_1}, ..., U_{\lambda_n}\}$ covers $f(X)$. So $f(X)$ is compact. □

Proposition 5.11. Let X_1 and X_2 be topological spaces, $A \subseteq X_1$, and $f : A \to X_2$ be a continuous map. If A is compact in X_1 then $f(A)$ is compact in X_2.

Proof. The proof of this is similar to Proposition 5.10. □

Remark 5.12 (Extreme Value Theorem and Compact Spaces)**.** One incredible result that follows from these propositions is the Extreme Value Theorem: if $f : X \to \mathbb{R}$ is a real valued, continuous function from a compact space X to the real numbers $\mathbb{R}$, then there is an $x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$. The proof is as follows:

Proof. The proof of this refers to the Heine-Borel Theorem, which is proven later in this section. Since X is compact, it follows that the image $f(X)$ is compact in $\mathbb{R}$. But by the Heine-Borel Theorem (Theorem **??**), it follows that $f(X)$ is closed and bounded. By the Completeness axiom there exists an upper bound of $f(X)$. Denote this upper bound $M \in f(X)$. By definition of the supremum, M is a limit point of $f(X)$, and as $f(X)$ is closed and therefore must contain all of its limit points, $M \in f(X)$. So, there must exist an element $x \in X$ such that $f(x) = M$.

□

5.2.1 Compactness of $[a, b]$

Theorem 5.13. The real line $\mathbb{R}$ is not compact.

Proof. Let $\mathbb{R}$ be a metric space equipped with the usual euclidean metric. Consider that, for a metric space to be compact, every open cover must admit a finite subcover. To prove that $\mathbb{R}$ is not compact, we will use proof by contradiction. Consider the open cover $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{N}\}$. Seeing that this is an open cover is trivial. Now suppose that there is a finite subcover of $\mathcal{C}$. That is, there exists $N \in \mathbb{N}$ such that $(-N, N)$ contains every other element of $\mathcal{C}$. But take any $x > N$. Clearly $x \in \mathcal{C}$, as $\mathcal{C}$ is a cover of $\mathbb{R}$. However, $x \notin (-N, N)$, and hence we have a contradiction. So, $\mathbb{R}$ is not compact. $\square$

Proposition 5.14. Any closed, bounded interval $[a, b]$ in $\mathbb{R}$ is compact.

Proof. Suppose that $[a, b]$ is not compact, then there exists an open cover $\mathcal{C}_\Lambda$ such that $[a, b] \subset \bigcup\limits_{\lambda \in \Lambda} U\lambda$ with no finite subcover. We can write $[a, b] = [a, m_1] \cup [m_1, b]$, where $m_1 = (a + b)/2$ (the midpoint of a and b). Consider that the union of two intervals with finite subcovers will itself have a finite subcover. Because there is no finite subcover for $[a, b]$, then for at least one of $[a, m_1]$ or $[m_1, b]$, there is no finite subcover for the interval.

Now pick whichever interval does not have a finite subcover (or either one if both do not). Suppose this interval is $[m_1, b]$. Again, dividing the interval in half, by the same logic above at least one of the subintervals will not have a finite subcover. We continue this process to obtain a sequence of **closed, bounded, and nested** intervals:

$$[a, b] \supset I_1 \supset I_2 \supset I_3 ...$$

Lemma 5.15. Consider that the intersection of closed, bounded nested intervals is non-empty

Proof. Each closed, bounded interval has a minimum and a maximum. Let (m_n) be the sequence of minima for the nested intervals, and (M_n) the sequence of maxima for the nested intervals. Observe that $m_n < M_n$ for all $n \in \mathbb{N}$. Therefore $\lim\limits_{n\to\infty} m_n \leq \lim\limits_{n\to\infty} M_n$. So either $\lim\limits_{n\to\infty} m_n \neq \lim\limits_{n\to\infty} M_n$, in which case there exists an interval $[m, M]$ contained in every interval I_n for all $n \in \mathbb{N}$. Or, $\lim\limits_{n\to\infty} m_n = \lim\limits_{n\to\infty} M_n = x$, in which case there exists an x contained in every interval I_n for all $n \in \mathbb{N}$. In either case, the intersection of all closed, bounded nested intervals is non-empty. $\square$

But, there must exist an open set in our open cover containing x, U_x, by assumption. In $\mathbb{R}$, open sets are the unions of open intervals. Therefore U_x must have some open interval (μ_1, μ_2) containing x within it.

Consider this open interval (μ_1, μ_2). We wish to show that some I_k is contained within (μ_1, μ_2), which would be a contradiction as this would imply that the singleton set U_x is a finite subcover for the intervals $I_k, I_{k+1}, ...$ Consider that by Lemma 5.15, there exists some x in all I_n, $n \in \mathbb{N}$. Take any interval I_k, $k \in \mathbb{N}$ such that the length of the interval $|I_k| < q/2$, where $q = \max\{x - \mu_1, \mu_2 - x\}$. That is,

$$\frac{b-a}{2^k} < q$$

$$\Rightarrow \frac{b-a}{q} < 2^k$$

$$\Rightarrow k > \log_2\left(\frac{b-a}{q}\right)$$

Then we are guaranteed to have an interval I_k contained within $[\mu_1, \mu_2]$. Then U_x is a finite subcover for all I_j, $k \leq j \in \mathbb{N}$. Hence, we have a contradiction. $\square$

5.2.2 Bolzano-Weierstrass Property and Compactness

We first recall the Bolzano-Weierstrass Theorem:

Theorem 5.16. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

We define the Bolzano-Weierstrass Property to be the following:

Definition 5.17. A set U in a metric space has the **Bolzano-Weierstrass property** if every sequence in U has a convergent subsequence.

5.3 Compactness for Metric Spaces

Theorem 5.18. Lebesgue Number Let X be a compact metric space and let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X. Then there exists a positive number $\delta > 0$ known as the Lebesgue Number such that for all x in X, $B_\delta(x)$ lies entirely inside some U_λ.

5.3.1 Uniform Continuity

Definition 5.19. A map $f : X_1 \to X_2$ of metric spaces with metrics d_1 and d_2 is uniformly continuous on X_1 if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), f(y)) < \epsilon$ for any x, y in X_1 satisfying $d_1(x, y) < \delta$.

Note that uniform continuity is a stronger notion than continuity in metric spaces, which is defined in Definition 3.26.Note the order of quantifiers:

$$\begin{aligned}\textbf{Continuity:}\ & (\forall \epsilon > 0)\ (\forall x \in X_1)\ (\exists \delta > 0)\ (\forall x_0 \in X_1),\\ & d_{X_1}(x, x_0) < \delta \Rightarrow d_{X_2}(f(x), f(x_0)) < \epsilon\\ \textbf{Uniform Continuity:}\ & (\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in X_1)\ (\forall x_0 \in X_1),\\ & d_{X_1}(x, x_0) < \delta \Rightarrow d_{X_2}(f(x), f(x_0)) < \epsilon\end{aligned}$$

The following example demonstrates this.

Example 5.20. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, where both $\mathbb{R}$ are under the usual metric. f is continuous but not uniformly continuous.

Proof. Let $\epsilon > 0$. We can take $\delta = \min\left\{1, \frac{\epsilon}{1+2|x_0|}\right\}$. Assuming $|x - x_0| < \delta$, then we have that

$$|x^2 - x_0^2| = |x - x_0||x + x_0|$$

But if $|x - x_0| \leq 1$, then $-1 \leq x - x_0 \leq 1 \Rightarrow -1 + 2x_0 \leq x + x_0 \leq 1 + 2x_0$, and so $|x + x_0| \leq 1 + 2|x_0|$. So,

$$\begin{aligned}|x^2 - x_0^2| \leq & |x - x_0|(1 + 2|x_0|)\\ < & \delta(1 + 2x_0) = \epsilon\end{aligned}$$

and so f is continuous. However, f is not uniformly continuous. To prove this, suppose that f is uniformly continuous. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that, for all $x, x_0 \in \mathbb{R}$,

$$|x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \epsilon.$$

Consider $\epsilon = 1$. If such a δ existed and $x_0 = x + \delta$, then we would have that

$$\begin{aligned}|x^2 - (x + \delta)^2| < & 1\\ \Rightarrow |2x\delta + \delta^2| < & 1\end{aligned}$$

Which is a contradiction, as we can choose x to be arbitrarily large. So f is not uniformly continuous. $\square$

Proposition 5.21. If $f : X_1 \to X_2$ is a continuous map of metric spaces and if X_1 is compact, then f is uniformly continuous on X_1.

Proof. Let $\epsilon > 0$. As f is continuous for each x in X_1, there exists $\delta(x) > 0$ such that $d_2(f(x), f(y)) < \frac{1}{2}\epsilon$ for all y satisfying $d_1(x, y) < 2\delta(x)$. The collection $\{B_{\delta(x)}(x) \mid x \in X_1\}$ is an open cover for X_1. By the compactness of X_1, there is a finite subcover of $\{B_{\delta(x)}(x) \mid x \in X_1\}$, $\{B_{\delta(x)}(x_1), B_{\delta(x_2)}(x), ..., B_{\delta(x)}(x_n)\}$. Let $\delta = \min\{\delta(x_1), \delta(x_2), ..., \delta(x_n)\}$. Given any $x, y \in X_1$ satisfying $d_1(x, y) < \delta$, (1) there is some i in $1, 2, ..., r$ such that $d(x, x_i) < \delta(x_i)$, and then (2) $d_1(y, x_i) \leq d_1(y, x) + d_1(x, x_i) < \delta + \delta(x_i) < 2\delta$.

Now by (1), $d_2(f(x), f(x_i)) < \frac{1}{2}\epsilon$, and by (2), $d_2(f(y), f(x_i) < \frac{1}{2}\epsilon$, and so

$$d_2(f(x), f(y)) \leq d_2(f(x), f(x_i)) + d_2(f(x_i), f(y)) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

as required.

□

Remark 5.22. Compactness is NOT a necessary condition for uniform continuity. Consider any metric space X_1 and the identity map $i : X_1 \to X_1$. This is a uniformly continuous map that does not depend on X_1 being compact.

6 Fourier Series